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1998 J. Phys. A: Math. Gen. 31 1025

(http://iopscience.iop.org/0305-4470/31/3/013)

J. Phys. A: Math. Gen. **31** (1998) 1025–1033. Printed in the UK PII: S0305-4470(98)84767-7

Compatible Jacobi manifolds: geometry and reduction

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Received 6 October 1997

Abstract. We give a generalization of bi-Hamiltonian manifolds, extending the notion of compatible Poisson tensors to Jacobi structures. We present some necessary and sufficient conditions for two Jacobi structures defined on the same smooth manifold to be compatible. We establish the relationship between the compatibility and the conformal equivalence of two Jacobi manifolds. We study the reduction of compatible Jacobi manifolds.

Introduction

The idea of endowing a differentiable manifold M with two 'compatible' structures is due to Magri [8], when he considered two Poisson tensors, Λ_1 and Λ_2, on M, verifying the condition $[\Lambda_1, \Lambda_2] = 0$. A manifold equipped with two compatible Poisson tensors is a bi-Hamiltonian manifold or a Poisson–Nijenhuis manifold [9]. The bi-Hamiltonian structures on manifolds play an important role in the study of integrable systems, see for example [1, 14, 15].

The notion of Jacobi manifold was introduced by Lichnerowicz [7] in 1978, and it includes the concepts of symplectic, Poisson, contact and co-symplectic manifolds. The Jacobi manifolds are then a very rich geometrical tool‡.

A *Jacobi manifold* is a triple (M, C, E) where M is a differentiable manifold, C is a 2-times contravariant skew-symmetric tensor field on M and E is a vector field on M, such that $[E, C] = 0$ and $[C, C] = 2E \wedge C$, where $[,]$ is the *Schouten–Nijenhuis* bracket. When $E = 0$, the Jacobi manifold is a Poisson manifold.

If (M, C, E) is a Jacobi manifold, the *Jacobi bracket* of $f, g \in C^\infty(M, \mathbb{R})$, is given by $\{f, g\} = C(df, dg) + f(E.g) - g(E.f)$. The Jacobi bracket defines a structure of *local Lie algebra* [4, 6] on $C^\infty(M, \mathbb{R})$.

In this paper we extend the notion of compatible Poisson tensors to the case of two Jacobi structures defined on a differentiable manifold. Some properties of compatible Jacobi structures are studied.

The reduction of Jacobi manifolds was established, from a geometric point of view, in [11] and [12]. Using the Jacobi reduction, we deduce a reduction theorem for compatible Jacobi manifolds.

The paper is divided into three sections. In section 1, we introduce the concept of compatibility of two Jacobi structures defined on the same differentiable manifold, and we give the necessary and sufficient conditions for two Jacobi manifolds to be compatible.

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‡ In a recent paper, de León *et al* [2] studied the Lichnerowicz–Jacobi cohomology of Jacobi manifolds.

We establish the relationship between the compatibility of two Jacobi manifolds and the compatibility of the corresponding associated homogeneous Poisson manifolds. In section 2, we prove some results involving the compatibility and the conformal equivalence of two Jacobi manifolds. In section 3, we study the reduction of compatible Jacobi manifolds. Finally, in a note added in proof, we compare the algebraic perspective of the Jacobi reduction that appears in a recent paper of Ibort *et al* [5], which envolves a Lie subalgebra and an ideal of $C^\infty(M, \mathbb{R})$, with the geometric procedure, that is based on the Marsden–Ratiu Poisson reduction [10]. We show that the geometric procedure can be obtained from the algebraic one, by a suitable choice of a Lie subalgebra and an ideal of $C^\infty(M, \mathbb{R})$. We prove that both Jacobi reductions can be applied in the case of two compatible Jacobi structures, generating compatible reduced Jacobi manifolds.

All manifolds, maps, vector and tensor fields are assumed to be differentiable of class C^∞ and we use the conventions of [7] in the definition of the Schouten–Nijenhuis bracket.

1. Compatible Jacobi manifolds

Let M be a differentiable manifold equipped with two Jacobi structures, (C_1, E_1) and (C_2, E_2).

Definition 1.1. Two Jacobi structures (C_1, E_1) and (C_2, E_2) on M are said to be *compatible* if $(C_1 + C_2, E_1 + E_2)$ is again a Jacobi structure on M. Under these conditions, the Jacobi manifolds (M, C_1, E_1) and (M, C_2, E_2) are said to be *compatible*.

Remark 1.2. Let us denote by $\{,\}_i$ the Jacobi bracket on M corresponding to (C_i, E_i), $i = 1, 2$. From definition 1.1, we may conclude that $\{,\} = \{,\}_1 + \{,\}_2$ is a Jacobi bracket on M if $\{,\}_1$ and $\{,\}_2$ are compatible.

The following theorem states the necessary and sufficient conditions for two Jacobi structures, defined on the same differentiable manifold, to be compatible.

Theorem 1.3. The Jacobi manifolds (M, C_1, E_1) and (M, C_2, E_2) are compatible if and only if

(i) $$[C_1, C_2] = E_1 \wedge C_2 + E_2 \wedge C_1$$

(ii) $$[E_1, C_2] + [E_2, C_1] = 0.$$

Proof. Suppose that conditions (i) and (ii) hold. Then

$$[E_1 + E_2, C_1 + C_2] = [E_1, C_1] + [E_1, C_2] + [E_2, C_1] + [E_2, C_2] = 0$$

$$\begin{aligned}[C_1 + C_2, C_1 + C_2] &= [C_1, C_1] + 2[C_1, C_2] + [C_2, C_2] \\ &= 2(E_1 + E_2) \wedge (C_1 + C_2)\end{aligned}$$

and $(M, C_1 + C_2, E_1 + E_2)$ is a Jacobi manifold.

Now, suppose that $(C_1 + C_2, E_1 + E_2)$ is a Jacobi structure on M. Since

$$[E_1 + E_2, C_1 + C_2] = [E_1, C_2] + [E_2, C_1]$$

condition (ii) holds.

We have

$$[C_1 + C_2, C_1 + C_2] = 2E_1 \wedge C_1 + 2[C_1, C_2] + 2E_2 \wedge C_2. \tag{1}$$

On the other hand

$$[C_1 + C_2, C_1 + C_2] = 2(E_1 + E_2) \wedge (C_1 + C_2). \tag{2}$$

From equations (1) and (2) we obtain condition (i). □

Corollary 1.4. If (C_1, E_1) and (C_2, E_2) are compatible Jacobi structures on M, then for any $\lambda \in \mathbb{R}$

$$(C_1 + \lambda C_2, E_1 + \lambda E_2) = (C_\lambda, E_\lambda)$$

is a Jacobi structure on M, which is compatible with (C_i, E_i), $i = 1, 2$.

Proof. Using theorem 1.3, a simple calculation gives

$$[E_\lambda, C_\lambda] = 0 \qquad [C_\lambda, C_\lambda] = 2E_\lambda \wedge C_\lambda \tag{3}$$

and

$$[C_\lambda, C_i] = E_\lambda \wedge C_i + E_i \wedge C_\lambda \qquad [E_\lambda, C_i] + [E_i, C_\lambda] = 0 \quad i = 1, 2. \tag{4}$$

By equation (3), (C_λ, E_λ) is a Jacobi structure on M while (4) ensures the compatibility of (C_λ, E_λ) with (C_i, E_i), $i = 1, 2$. □

What follows is a simple example of compatible Jacobi manifolds.

Example 1.5. Let M be a differentiable manifold of dimension $2n+1$ and let $(x_0, x_1, \ldots, x_n)$ be local coordinates on M. The two Jacobi structures (E_1, C_1) and (E_2, C_2) on M, where

$$E_1 = \frac{\partial}{\partial x_0} \quad \text{and} \quad C_1 = \frac{\partial}{\partial x_0} \wedge \left(\sum_{i=1}^{n} x_{2i-1} \frac{\partial}{\partial x_{2i-1}} \right) - \sum_{i=1}^{n} \frac{\partial}{\partial x_{2i-1}} \wedge \frac{\partial}{\partial x_{2i}}$$

$$E_2 = -\frac{\partial}{\partial x_2} \qquad \text{and} \qquad C_2 = x_1 C_1$$

are compatible.

Recall that if (M, C, E) is a Jacobi manifold, we may associate with each function $f \in C^\infty(M, \mathbb{R})$ a vector field on M, $X_f = [C, f] + fE$, which is called the *Hamiltonian vector field* of f. For all $f, g \in C^\infty(M, \mathbb{R})$, we have $[X_f, X_g] = X_{\{f,g\}}$ and $\{f, g\} = X_f.g - g(E.f)$.

As in the case of bi-Hamiltonian manifolds, the compatibility of two Jacobi structures defined on the same differentiable manifold can be stated using some properties that are equivalent to definition 1.1.

Theorem 1.6. Let (M, C_1, E_1) and (M, C_2, E_2) be two Jacobi manifolds, whose Jacobi brackets are denoted respectively by $\{,\}_1$ and $\{,\}_2$. Given a function $f \in C^\infty(M, \mathbb{R})$, we denote by X_f and Y_f the Hamiltonian vector fields of f with respect to the Jacobi structures (C_1, E_1) and (C_2, E_2), respectively.

The following properties 1–4 are equivalent:

1. (C_1, E_1) and (C_2, E_2) are compatible;
2. for all $f, g, h \in C^\infty(M, \mathbb{R})$

$$S_{\text{circ}}(\{f, \{g, h\}_1\}_2 + \{f, \{g, h\}_2\}_1) = 0$$

where S_{circ} means summmation after circular permutation;

3. for all $f, g \in C^\infty(M, \mathbb{R})$

$$[X_f, Y_g] + [Y_f, X_g] - X_{\{f,g\}_2} - Y_{\{f,g\}_1} = 0;$$

4. for all $f \in C^\infty(M, \mathbb{R})$
   (a) $\mathcal{L}(Y_f)C_1 + \mathcal{L}(X_f)C_2 + (E_2.f)C_1 + (E_1.f)C_2 = 0$
   (b) $[X_f, E_2] + [Y_f, E_1] + X_{(E_2.f)} + Y_{(E_1.f)} = 0.$

Proof. Let us suppose that condition 1 holds. Then, $(C, E) = (C_1 + C_2, E_1 + E_2)$ is a Jacobi structure on M and, as observed in remark 1.2, its Jacobi bracket is given by $\{,\} = \{,\}_1 + \{,\}_2$. For all $f, g, h \in C^\infty(M, \mathbb{R})$

$$S_{\mathrm{circ}}(\{f, \{g, h\}\}) = S_{\mathrm{circ}}(\{f, \{g, h\}_1\}_1 + \{f, \{g, h\}_1\}_2 + \{f, \{g, h\}_2\}_1 + \{f, \{g, h\}_2\}_2)$$

and the Jacobi identity for $\{,\}$, $\{,\}_1$ and $\{,\}_2$ gives condition 2.

Let us now suppose that condition 2 holds. For any $f, g, h \in C^\infty(M, \mathbb{R})$, we have

$$\begin{aligned}([E_1, C_2] + [E_2, C_1])(df, dg) &= E_1.(C_2(df, dg)) - C_2(df, d(E_1.g)) + C_2(dg, d(E_1.f))\\ &\quad + E_2.(C_1(df, dg)) - C_1(df, d(E_2.g)) + C_1(dg, d(E_2.f))\\ &= E_1.\{f, g\}_2 + E_2.\{f, g\}_1 - Y_f.(E_1.g)\\ &\quad + Y_g.(E_1.f) - X_f.(E_2.g) + X_g.(E_2.f) = 0 \end{aligned} \tag{5}$$

the last equality being obtained from condition 2 with $h = 1$; also

$$\begin{aligned}&([C_1, C_2] - E_1 \wedge C_2 - E_2 \wedge C_1)(df, dg, dh)\\ &\quad = S_{\mathrm{circ}}(C_2(df, d(C_1(dg, dh))) + C_1(df, d(C_2(dg, dh))) - (E_1.f)C_2(dg, dh)\\ &\qquad - (E_2.f)C_1(dg, dh))\\ &\quad = S_{\mathrm{circ}}(\{f, \{g, h\}_1\}_2 + \{f, \{g, h\}_2\}_1 - f(([E_1, C_2] + [E_2, C_1])(dg, dh))) = 0.\end{aligned} \tag{6}$$

Applying theorem 1.3, equations (5) and (6) ensure the compatibility of (C_1, E_1) and (C_2, E_2).

Now, we show the equivalence of conditions 2 and 3. Let us take any $h \in C^\infty(M, \mathbb{R})$. Then

$$\begin{aligned}&([X_f, Y_g] + [Y_f, X_g] - X_{\{f,g\}_2} - Y_{\{f,g\}_1}).h\\ &\quad = X_f.(\{g, h\}_2 + (E_2.g)h) - Y_g.(\{f, h\}_1 + (E_1.f)h) + Y_f.(\{g, h\}_1 + (E_1.g)h)\\ &\qquad - X_g.(\{f, h\}_2 + (E_2.f)h) - \{\{f, g\}_2, h\}_1 - (E_1.\{f, g\}_2)h - \{\{f, g\}_1, h\}_2\\ &\qquad - (E_2.\{f, g\}_1)h\\ &\quad = S_{\mathrm{circ}}(\{f, \{g, h\}_2\}_1 + \{f, \{g, h\}_1\}_2) + h(X_f.(E_2.g) - Y_g.(E_1.f)\\ &\qquad + Y_f.(E_1.g) - X_g.(E_2.f) - E_1.\{f, g\}_2 - E_2.\{f, g\}_1).\end{aligned} \tag{7}$$

If condition 2 holds, the two first terms of (7) vanish and, as we have remarked above (in equation (5)), the last term of (7) also vanishes. Then, because $h \in C^\infty(M, \mathbb{R})$ is an arbitrary function, we obtain condition 3.

If condition 3 holds, then equation (7) gives

$$\begin{aligned}0 &= S_{\mathrm{circ}}(\{f, \{g, h\}_2\}_1 + \{f, \{g, h\}_1\}_2) + h(X_f.(E_2.g) - Y_g.(E_1.f)\\ &\quad + Y_f.(E_1.g) - X_g.(E_2.f) - E_1.\{f, g\}_2 - E_2.\{f, g\}_1)\\ &= S_{\mathrm{circ}}(\{f, \{g, h\}_2\}_1 + \{f, \{g, h\}_1\}_2)\\ &\quad + h([E_1, Y_f] + [E_2, X_f] - X_{E_2.f} - Y_{E_1.f}).g.\end{aligned} \tag{8}$$

However, from condition 3, we can easily show that the last term of the right-hand side of (8) vanishes. Thus, equation (8) gives condition 2.

Finally, let us now show the equivalence of conditions 3 and 4. Take any $g \in C^\infty(M, \mathbb{R})$. Then

$$\begin{aligned}
&[\mathcal{L}(Y_f)C_1 + \mathcal{L}(X_f)C_2 + (E_2.f)C_1 + (E_1.f)C_2, g] \\
&\quad = [Y_f, X_g] - [Y_f, gE_1] + [X_f, Y_g] - [X_f, gE_2] - [C_1, \{f, g\}_2] \\
&\qquad - [C_1, (E_2.f)g] - [C_2, \{f, g\}_1] - [C_2, (E_1.f)g] + [(E_2.f)C_1, g] \\
&\qquad + [(E_1.f)C_2, g] \\
&\quad = [Y_f, X_g] + [X_f, Y_g] - X_{\{f,g\}_2} - Y_{\{f,g\}_1} - g(X_{E_2.f} + Y_{E_1.f} \\
&\qquad + [Y_f, E_1] + [X_f, E_2])
\end{aligned}$$

and, if condition 4 holds, we obtain condition 3.

Now, if condition 3 holds, taking $g = 1$, we imediately obtain condition 4(b). To prove that $3 \Rightarrow 4$(a), we take two arbitrary functions $g, h \in C^\infty(M, \mathbb{R})$ and we show that

$$(\mathcal{L}(Y_f)C_1 + \mathcal{L}(X_f)C_2 + (E_2.f)C_1 + (E_1.f)C_2)(dg, dh) = 0.$$

Using condition 2, which is equivalent to condition 3 as we have already proved, we obtain

$$\begin{aligned}
&(\mathcal{L}(Y_f)C_1 + \mathcal{L}(X_f)C_2 + (E_2.f)C_1 + (E_1.f)C_2)(dg, dh) \\
&\quad = Y_f.(C_1(dg, dh)) - C_1(dg, d(Y_f.h)) + C_1(dh, d(Y_f.g)) + X_f.(C_2(dg, dh)) \\
&\qquad - C_2(dg, d(X_f.h)) + C_2(dh, d(X_f.g)) + (E_2.f)C_1(dg, dh) \\
&\qquad + (E_1.f)C_2(dg, dh) \\
&\quad = g\{E_2.f, h\}_1 + g\{E_1.f, h\}_2 - h\{E_2.f, g\}_1 - h\{E_1.f, g\}_2 + \{(E_2.f)h, g\}_1 \\
&\qquad - \{(E_2.f)g, h\}_1 + 2(E_1.f)\{g, h\}_2 + \{(E_1.f)h, g\}_2 - \{(E_1.f)g, h\}_2 \\
&\qquad + 2(E_2.f)\{g, h\}_1 - (E_2.f)g(E_1.h) + (E_2.f)h(E_1.g) - (E_1.f)g(E_2.h) \\
&\qquad + (E_1.f)h(E_2.g) = 0.
\end{aligned}$$

□

Let (M, C_1, E_1) and (M, C_2, E_2) be two Jacobi manifolds. Let us take the corresponding associated homogeneous Poisson manifolds [7] (P, Λ_1) and (P, Λ_2), with $P = \mathbb{R} \times M$ and

$$\Lambda_i = \mathrm{e}^{-t}\left(C_i + \frac{\partial}{\partial t} \wedge E_i\right) \qquad i = 1, 2.$$

Proposition 1.7. The Jacobi manifolds (M, C_1, E_1) and (M, C_2, E_2) are compatible if and only if the Poisson tensors Λ_1 and Λ_2 on P are compatible.

Proof. Using the properties of the Schouten–Nijenhuis bracket, with $\Lambda_i = \mathrm{e}^{-t}\left(C_i + \frac{\partial}{\partial t} \wedge E_i\right)$, $i = 1, 2$, we obtain

$$[\Lambda_1, \Lambda_2] = \mathrm{e}^{-2t}(-E_1 \wedge C_2 - E_2 \wedge C_1 + [C_1, C_2] - \frac{\partial}{\partial t} \wedge ([E_1, C_2] + [E_2, C_1])).$$

So, $[\Lambda_1, \Lambda_2] = 0$ is equivalent to

$$-E_1 \wedge C_2 - E_2 \wedge C_1 + [C_1, C_2] = 0 \qquad \text{and} \qquad [E_1, C_2] + [E_2, C_1] = 0.$$

Applying theorem 1.3, we obtain the desired result. □

2. Compatibility and conformal equivalence of two Jacobi structures

Let us recall that if (M, C, E) is a Jacobi manifold, with the Jacobi bracket denoted by $\{,\}$ and $a \in C^\infty(M, \mathbb{R})$ is a differentiable function that never vanishes on M, we may define a new Jacobi bracket, setting $\{f, g\}_a = (1/a)\{af, ag\}$, for any $f, g \in C^\infty(M, \mathbb{R})$. The Jacobi structure on M associated with the new Jacobi bracket is the pair (C_a, E_a) , where $C_a = aC$ and $E_a = [C, a]+aE = X_a$. The two Jacobi structures (C, E) and (C_a, E_a) on M, are said to be *conformally equivalent*. Moreover, the property of conformality determines an equivalence relation among Jacobi structures on the same differentiable manifold. If (M, C, E) is a Jacobi manifold, the equivalence class of all Jacobi structures on M that are conformally equivalent to (C, E), is called the *conformal Jacobi structure* of M.

We now show the relationship between the compatibility of Jacobi manifolds and the conformal equivalence of their Jacobi structures.

Proposition 2.1. Let (M, C, E) be a Jacobi manifold and let $h, k \in C^\infty(M, \mathbb{R})$ be two distinct functions that never vanish on M. If (C_h, E_h) and (C_k, E_k) are two Jacobi structures on M, that are conformally equivalent to (C, E), then (C_h, E_h) and (C_k, E_k) are compatible, where $C_h = hC$, $C_k = kC$, $E_h = hE + [C, h]$ and $E_k = kE + [C, k]$.

In particular, any Jacobi structure on M which is conformally equivalent to (C, E), is compatible with (C, E).

Proof. We have

$$\begin{aligned}[C_h, C_k] - E_h \wedge C_k - E_k \wedge C_h &= h[k, C] \wedge C + k[C, h] \wedge C + kh[C, C] \\ &\quad - 2khE \wedge C - k[C, h] \wedge C - h[C, k] \wedge C = 0\end{aligned}$$

and also

$$\begin{aligned}[E_h, C_k] + [E_k, C_h] &= [hE, k] \wedge C + k[hE, C] + [[C, h], k] \wedge C + k[[C, h], C] \\ &\quad + [kE, h] \wedge C + h[kE, C] + [[C, k], h] \wedge C + h[[C, k], C] \\ &= h(E.k) \wedge C + k[C, h] \wedge E - k[h, E] \wedge C + kE \wedge [h, C] \\ &\quad + k(E.h)C + h[C, k] \wedge E - h[k, E] \wedge C + hE \wedge [k, C] = 0.\end{aligned}$$

By theorem 1.3 the proof is completed. □

Proposition 2.2. Let (M, C_1, E_1) and (M, C_2, E_2) be two compatible Jacobi manifolds. Then, for any function $h \in C^\infty(M, \mathbb{R})$ that never vanishes on M, the Jacobi manifolds (M, C_{1h}, E_{1h}) and (M, C_{2h}, E_{2h}), with $C_{ih} = hC_i$ and $E_{ih} = hE_i + [C_i, h]$, $i = 1, 2$, are compatible.

Proof. We have

$$\begin{aligned}[C_{1h}, C_{2h}] - C_{1h} \wedge E_{2h} - C_{2h} \wedge E_{1h} &= [hC_1, h] \wedge C_2 + h[hC_1, C_2] - h^2 C_1 \wedge E_2 \\ &\quad - hC_1 \wedge [C_2, h] - h^2 C_2 \wedge E_1 - hC_2 \wedge [C_1, h] \\ &= h^2([C_1, C_2] - C_1 \wedge E_2 - C_2 \wedge E_1)\end{aligned}$$

and, since (C_1, E_1) and (C_2, E_2) are compatible Jacobi structures, we obtain

$$[C_{1h}, C_{2h}] - C_{1h} \wedge E_{2h} - C_{2h} \wedge E_{1h} = 0. \tag{9}$$

On the other hand

$$
\begin{aligned}
[E_{2h}, C_{1h}] + [E_{1h}, C_{2h}] = h(E_2.h)C_1 &+ [hE_2, h]C_1 + h[hE_2, C_1] + [[C_2, h], h] \wedge C_1 \\
&+ h[[C_2, h], C_1] + [hE_1, h]C_2 + h[hE_1, C_2] \\
&+ [[C_1, h], h] \wedge C_2 + h[[C_1, h], C_2] \\
= h(E_2.h)C_1 &+ h[C_1, h] \wedge E_2 + [[C_2, h], h] \wedge C_1 - h[[C_1, C_2], h] \\
&+ h(E_1.h)C_2 + h[C_2, h] \wedge E_1 + [[C_1, h], h] \wedge C_2
\end{aligned}
$$

and, because $[C_1, C_2] = E_2 \wedge C_1 + E_1 \wedge C_2$, a straightfoward calculation leads to

$$[E_{2h}, C_{1h}] + [E_{1h}, C_{2h}] = 0. \qquad (10)$$

From equations (9) and (10), we obtain the desired result. □

3. Reduction of compatible Jacobi manifolds

The reduction theorem of Jacobi manifolds that appears in [11] and [12] is inspired on the Marsden–Ratiu Poisson reduction theorem [10]. It states the following.

Theorem 3.1. Let (M, C, E) be a Jacobi manifold, S a submanifold of M and D a vector sub-bundle of $T_S M$, which satisfy the following conditions:

(i) the distribution $TS \cap D$ on S is completely integrable; the set $\overline{S}$ of leaves of the foliation defined on S by that distribution is a differentiable manifold, and the canonical projection $\pi : S \to \overline{S}$ is a submersion;
(ii) for any $F, G \in C^\infty(M, \mathbb{R})$ with differentials dF and dG, restricted to S, vanishing on D, the differential $d\{F, G\}$ restricted to S vanishes on D;
(iii) if $D^0 \subset T^*_S M$ denotes the annihilator of D, then $C^\#(D^0) \subset TS + D$, and the restriction of E to S is a differentiable section of $TS + D$.

Then there exists on $\overline{S}$ a unique compatible Jacobi structure, $(\overline{C}, \overline{E})$ such that, for any $f, g \in C^\infty(\overline{S}, \mathbb{R})$ and any differentiable extensions F of $f \circ \pi$ and G of $g \circ \pi$ whose differentials dF and dG, restricted to S, vanish on D:

$$\{f, g\} \circ \pi = \{F, G\} \circ i \qquad (11)$$

where i is the canonical injection of S in M.

The last theorem can readily be extended to the case of a differentiable manifold equipped with two compatible Jacobi structures.

Theorem 3.2. Let M be a differentiable manifold equipped with two compatible Jacobi structures (C_1, E_1) and (C_2, E_2), S a submanifold of M and D a vector sub-bundle of $T_S M$ satisfying condition (i) of theorem 3.1. Let us suppose that conditions (ii) and (iii) of theorem 3.1 are verified for both Jacobi structures (C_1, E_1) and (C_2, E_2) on M. Then, there exists on $\overline{S}$ two unique compatible Jacobi structures, $(\overline{C}_1, \overline{E}_1)$ and $(\overline{C}_2, \overline{E}_2)$ with Jacobi brackets given by

$$\{f, g\}_j \circ \pi = \{F, G\}_j \circ i \qquad j = 1, 2$$

where $f, g \in C^\infty(\overline{S}, \mathbb{R})$ and $F, G \in C^\infty(M, \mathbb{R})$ are differentiable extensions of $f \circ \pi$ and of $g \circ \pi$, respectively, with differentials dF and dG, restricted to S, vanishing on D.

Proof. The existence and uniqueness of the reduced Jacobi structures on $\overline{S}$ are ensured by theorem 3.1. It only remains to show that these Jacobi structures on $\overline{S}$ are compatible.

Take any $f, g, h \in C^\infty(\overline{S}, \mathbb{R})$ and let G and H be differentiable extensions of $g \circ \pi$ and of $h \circ \pi$, respectively, whose differentials dG and dH, restricted to S vanish on D.

Then

$$\{g, h\}_1 \circ \pi = \{G, H\}_1 \circ i$$

which means that $\{G, H\} \in C^\infty(M, \mathbb{R})$ is an extension of $\{g, h\}_1 \circ \pi \in C^\infty(S, \mathbb{R})$ and by condition (i) the restriction of $d\{G, H\}$ to S vanishes on D. Thus

$$\{f, \{g, h\}_1\}_2 \circ \pi = \{F, \{G, H\}_1\}_2 \circ i$$

where F is a differentiable extension of $f \circ \pi$ such that the restriction of dF to S vanishes on D.

A similar computation leads to

$$\{f, \{g, h\}_2\}_1 \circ \pi = \{F, \{G, H\}_2\}_1 \circ i.$$

Therefore

$$S_{\mathrm{circ}}(\{f, \{g, h\}_1\}_2 \circ \pi + \{f, \{g, h\}_2\}_1 \circ \pi) = S_{\mathrm{circ}}(\{F, \{G, H\}_1\}_2 \circ i + \{F, \{G, H\}_2\}_1 \circ i) \tag{12}$$

and, since (C_1, E_1) and (C_2, E_2) are compatible Jacobi structures on M, the right member of (12) vanishes. Then

$$S_{\mathrm{circ}}(\{f, \{g, h\}_1\}_2 + \{f, \{g, h\}_2\}_1) = 0$$

and the Jacobi structures $(\overline{C}_1, \overline{E}_1)$ and $(\overline{C}_2, \overline{E}_2)$ on $\overline{S}$ are compatible. □

We remark that last theorem is a natural extension of the reduction theorem for bi-Hamiltonian manifolds [13].

Acknowledgments

The author wishes to thank the referees for helpful comments and suggestions. This work was partially supported under grant PRAXIS: 2/2.1/MAT/19/94; 2/2.1/MAT/458/94.

Note added in proof. In a recent paper, Ibort *et al* [5] studied the reduction of the algebra of functions defined on a Jacobi manifold, the so-called *Jacobi algebra*. In that work, given a Jacobi manifold (M, C, E), the main idea is to choose a subalgebra $\mathcal{A}$ of $(C^\infty(M, \mathbb{R}), .)$ which is also a Lie subalgebra of $(C^\infty(M, \mathbb{R}), \{,\})$, and an ideal $\mathcal{I}$ of $(C^\infty(M, \mathbb{R}), .)$ such that $\mathcal{A} \cap \mathcal{I}$ is an ideal with respect to the associative structure and is also an invariant Lie subalgebra. Then, the quotient space $\mathcal{A}/\mathcal{A} \cap \mathcal{I}$ is a subalgebra of $C^\infty(M, \mathbb{R})/\mathcal{I}$ and carries a (reduced) Lie algebra structure. The authors show that, if the ideal $\mathcal{I}$ is E-invariant (i.e. $\mathcal{L}(E)\mathcal{I} \subset \mathcal{I}$), its normalizer $\mathcal{N}_\mathcal{I}$ fullfils all the conditions imposed on $\mathcal{A}$ and so the quotient space $\mathcal{N}_\mathcal{I}/\mathcal{N}_\mathcal{I} \cap \mathcal{I}$ inherits a Jacobi algebra structure induced from that of $C^\infty(M, \mathbb{R})$. Moreover, if there exists a differentiable manifold N such that $C^\infty(N, \mathbb{R}) = \mathcal{N}_\mathcal{I}/\mathcal{N}_\mathcal{I} \cap \mathcal{I}$, then N inherits the structure of a Jacobi manifold and the bracket among the functions is given by the bracket of the Lie algebra structure induced in $\mathcal{N}_\mathcal{I}/\mathcal{N}_\mathcal{I} \cap \mathcal{I}$.

Using the techniques of [3], we can show that theorem 3.1 can be obtained from the reduction procedure of A Ibort, M de León and G Marmo's paper. Moreover, several particular cases in that paper, also appear in [12].

Let (M, C, E) be a Jacobi manifold, S a submanifold of M and D a vector sub-bundle of $T_S M$ such that conditions (i), (ii) and (iii) of theorem 3.1 hold. We take

$$\mathcal{A} = \{f \in C^\infty(M, \mathbb{R}) : df \in D^0\}.$$

Then, it is obvious that $\mathcal{A}$ is a subalgebra of $(C^\infty(M, \mathbb{R}), .)$ and, by condition (ii), $\mathcal{A}$ is also a Lie subalgebra of $(C^\infty(M, \mathbb{R}), \{,\})$.

We choose the following ideal for $(C^\infty(M,\mathbb{R}), .)$:

$$\mathcal{I} = \{f \in C^\infty(M,\mathbb{R}) : f\,|_S = 0\}.$$

Then, by condition (i), the quotient space $\mathcal{A}/\mathcal{A}\cap\mathcal{I}$ can be identified with $C^\infty(\overline{S},\mathbb{R})$.

Now, we have to prove that $\mathcal{A}\cap\mathcal{I}$ is an invariant Lie subalgebra of $\mathcal{A}$; that is, we have to show that $\{\mathcal{A}\cap\mathcal{I},\mathcal{A}\}\subset\mathcal{A}\cap\mathcal{I}$. Let $f\in\mathcal{A}$ and $g\in\mathcal{A}\cap\mathcal{I}$. Since $\mathcal{A}$ is a Lie subalgebra of $(C^\infty(M,\mathbb{R}),\{,\})$, $\{f,g\}\in\mathcal{A}$. For all $x\in S$, $g(x)=0$ and so

$$\{f,g\}(x) = (X_f.g)(x). \tag{13}$$

However, if $g\in\mathcal{A}\cap\mathcal{I}$, then $dg(x)\in(D_x+T_xS)^0$, for all $x\in S$. So, by (13), the bracket $\{f,g\}(x)$ vanishes on S if $X_f(x)\in D_x+T_xS$, for all $x\in S$, which is just condition (iii) of theorem 3.1.

Finally, we remark that, for each $f\in\mathcal{A}$, the operator D_f given by

$$D_f(g) = \{f,g\} \qquad g\in\mathcal{A}$$

is a differential operator of order one and that $\mathcal{A}\cap\mathcal{I}$ is invariant for each D_f, $f\in\mathcal{A}$. Thus, D_f induces a differential operator of order one in the quotient Lie algebra $\mathcal{A}/\mathcal{A}\cap\mathcal{I}$, and we obtain a Jacobi algebra structure in $\mathcal{A}/\mathcal{A}\cap\mathcal{I}$. Note that the Jacobi bracket in $\overline{S}$, given by (11) is just the Lie bracket on $\mathcal{A}/\mathcal{A}\cap\mathcal{I}$.

We may now apply this result to the case of compatible Jacobi manifolds.

Let us take a differentiable manifold M equipped with two compatible Jacobi structures (C_1,E_1) and (C_2,E_2), S a submanifold of M and D a vector sub-bundle of T_SM verifying the conditions of theorem 3.2. The subalgebra $\mathcal{A}=\{f\in C^\infty(M,\mathbb{R}) : df\in D^0\}$ has now two Lie algebra structures, $(\mathcal{A},\{,\}_1)$ and $(\mathcal{A},\{,\}_2)$. The previous discussion allow us to conclude that the quotient space $\mathcal{A}/\mathcal{A}\cap\mathcal{I}=C^\infty(\overline{S},\mathbb{R})$, where $\mathcal{I}=\{f\in C^\infty(M,\mathbb{R}) : f\,|_S=0\}$, inherits two (reduced) Jacobi structures. Using the same technical arguments of the proof of theorem 3.2, we may show that the two Jacobi structures on $C^\infty(\overline{S},\mathbb{R})=\mathcal{A}/\mathcal{A}\cap\mathcal{I}$ are compatible.

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